# On estimation of the Orey index for a class of Gaussian processes.

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**Abstract**

Orey suggested the definition of some index for Gaussian processes with stationary increments which determines various properties of the sample paths of this process. We give an extension of the definition of the Orey index for a second order stochastic processes which may not have stationary increments and estimate the Orey index for Gaussian process from discrete observations of its sample paths.

*Keywords*: Gaussian process, Hurst index, fractional Brownian motion, incremental variance function

## 1 Introduction

The fractional Brownian motion (fBm) is a popular model in financial mathematics, economics and natural sciences. As is well known the fBm $B^H$ is the only continuous Gaussian process which is selfsimilar with stationary increments and depending on index $0 < H < 1$. Moreover, a fBm with Hurst index $H$ is Hölder up to order $H$.

For a real mean zero Gaussian process with stationary increments, Orey suggested the following definition of index.

**Definition 1 (see [10], [8])** *Let $X$ be a real-valued mean zero Gaussian stochastic process with stationary increments and continuous in quadratic mean. Let $\sigma_X$ be the incremental variance of $X$ given by $\sigma_X^2(h) = \mathbf{E}[X(t+h) - X(t)]^2$ for $t, h \geqslant 0$. Define*

$$\widehat{\beta}_* := \inf\left\{\beta > 0\colon\ \lim_{h\downarrow 0} \frac{h^\beta}{\sigma_X(h)} = 0\right\} = \limsup_{h\downarrow 0} \frac{\ln \sigma_X(h)}{\ln h} \tag{1}$$

*and*

$$\widehat{\beta}^* := \sup\left\{\beta > 0\colon\ \lim_{h\downarrow 0} \frac{h^\beta}{\sigma_X(h)} = +\infty\right\} = \liminf_{h\downarrow 0} \frac{\ln \sigma_X(h)}{\ln h}\,. \tag{2}$$

*If $\widehat{\beta}_* = \widehat{\beta}^*$ then $X$ has the Orey index $\beta_X$.*

If Gaussian process with stationary increments has Orey index then almost all sample paths satisfy a Hölder condition of order $\gamma$ for each $\gamma \in (0, \beta_X)$ (see Section 9.4 of Cramer and Leadbetter [5]). For fBm $B^H$ with the Hurst index $0 < H < 1$ the Orey index $\beta_X = H$. So we have a class of Gaussian processes with stationary increments depending on Orey index $\beta_X$.

Recently there haves been two extensions of fBm which preserve many properties of fBm, but have no stationary increments except for particular parameter values. One of them is a so called sub-fractional Brownian motion (sfBm) and another one is a bifractional Brownian motion (bifBm). Thus it is very natural to extend the definition of the Orey index for Gaussian processes such that there was a possibility to consider processes which may not have stationary increments and are Hölder up to the Orey index.

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We shall give such extension of the Orey index. As it will be proved, processes sfBm and bifBm satisfy this extended definition of the Orey index and they are Hölder up to the Orey index. Moreover, for fBm, sfBm, and bifBm, the Orey index coincides with their self-similarity parameter. Therefore it is enough to construct and consider the asymptotic behavior of an estimate of the Orey index instead of estimating parameters of each of the processes under consideration.

Many authors have already considered the asymptotic behavior of the first- and second-order quadratic variations of Gaussian processes. The conditions in these papers were expressed in terms of covariance of a Gaussian process and depended on some parameter $\gamma \in (0, 2)$. If a Gaussian process has the Orey index then conditions on a covariance function may expressed by means of it. As it will be shown below, the Orey index can be obtained for some well-known Gaussian processes. Moreover, if we wanted to consider stochastic differential equations (SDE) driven by processes with a bounded $p$-variation, we should know when the Riemann-Stieltjes (RS) integral is defined. For Gaussian processes the Orey index helps to obtain these conditions.

The purpose of this paper is to give an extension of the definition of the Orey index for a second order stochastic processes which may not have stationary increments and to estimate the Orey index for Gaussian process from discrete observations of its sample paths.

Norvaiša [9] extends the definition of the Orey index for a second order stochastic processes which may not have stationary increments. He showed that sfBm and bifBm satisfies this extended definition of the Orey index. In this paper we shall give a different extension of the definition of the Orey index. This new definition will be more convenient for our purposes.

The paper is organized in the following way. Section 2 contains the definition of the Orey index for the second order stochastic process. The conditions when the second order stochastic process has the Orey index are also given. For some well-known Gaussian processes which do not have stationary increments the Orey index is obtained. Section 3 contains the results on an almost sure asymptotic behavior of the second-order quadratic variations of a Gaussian process. Here we also verify obtained conditions for the same well-known Gaussian processes.

# 2 Orey index for the second order stochastic processes

Let $X = \{X(t)\colon\ t \in [0, T]\}$ be a second order stochastic process with the incremental variance function $\sigma_X^2$ defined on $[0, T]^2 := [0, T] \times [0, T]$ with values

$$\sigma_X^2(s,t) := \mathbf{E}[X(t) - X(s)]^2, \quad (s,t) \in [0,T]^2.$$

Denote by $\Psi$ a class of continuous functions $\varphi\colon\ (0, T] \to [0, \infty)$ such that $\lim_{h\downarrow 0} \varphi(h) = 0$ and $\lim_{h\downarrow 0}[h \cdot L^3(h)] = 0$, where $L(h) = \varphi(h)/h \to \infty$, $h \downarrow 0$. For example, we can take $\varphi(h) = h \cdot |\ln h|^\alpha$ or $\varphi(h) = h^{1-\beta}$, where $\alpha > 0$, $0 < \beta < 1/3$. Set

$$\gamma_* := \inf\left\{\gamma > 0\colon\ \lim_{h\downarrow 0} \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s, s+h)} = 0\right\}, \tag{3}$$

$$\widetilde{\gamma}_* := \inf\left\{\gamma > 0\colon\ \lim_{h\downarrow 0} \frac{h^\gamma}{\sigma_X(0,h)} = 0\right\} \tag{4}$$

and

$$\gamma^* := \sup\left\{\gamma > 0\colon\ \lim_{h\downarrow 0} \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s, s+h)} = +\infty\right\}, \tag{5}$$

$$\widetilde{\gamma}^* := \sup\left\{\gamma > 0\colon\ \lim_{h\downarrow 0} \frac{h^\gamma}{\sigma_X(0,h)} = +\infty\right\}, \tag{6}$$

where $\varphi \in \Psi$. Note that $0 \leqslant \widetilde{\gamma}^* \leqslant \widetilde{\gamma}_* \leqslant +\infty$ and $0 \leqslant \gamma^* \leqslant \gamma_* \leqslant +\infty$.

We give the following extension of the Orey index.

**Definition 2** *Let $X = \{X(t)\colon\ t \in [0,T]\}$ be a second order stochastic process with the incremental variance function $\sigma_X^2$ such that $\sup_{0\leqslant s\leqslant T-h} \sigma_X(s, s+h) \to 0$ as $h \to 0$. If $\gamma_* = \widetilde{\gamma}_* = \gamma^* = \widetilde{\gamma}^*$ for any function $\varphi \in \Psi$, then we say that the process $X$ has the Orey index $\gamma_X = \gamma_* = \widetilde{\gamma}_* = \gamma^* = \widetilde{\gamma}^*$.*

**Remark 3** *From our definition of the Orey index we get the definition of the Orey index for a real-valued mean zero Gaussian stochastic process with stationary increments and continuous in quadratic mean.*

Let us introduce notions

$$\widehat{\gamma}_* := \limsup_{h\downarrow 0} \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} \quad \text{and} \quad \overline{\gamma}_* := \limsup_{h\downarrow 0} \frac{\ln \sigma_X(0,h)}{\ln h}, \tag{7}$$

$$\widehat{\gamma}^* := \liminf_{h\downarrow 0} \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} \quad \text{and} \quad \overline{\gamma}^* := \liminf_{h\downarrow 0} \frac{\ln \sigma_X(0,h)}{\ln h}. \tag{8}$$

We have that $\widetilde{\gamma}_* = \overline{\gamma}_*$ $\widetilde{\gamma}^* = \overline{\gamma}^*$. It follows from Remark 3 and (1) and (2). Now we compare quantities $\widehat{\gamma}^*$ and $\widehat{\gamma}_*$ with $\gamma^*$ and $\gamma_*$, respectively, for a second order stochastic process $X$.

**Lemma 4** *Let $X = \{X(t)\colon\ t \in [0,T]\}$ be a second order stochastic process with the incremental variance function $\sigma_X^2$ such that*

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h) \longrightarrow 0 \qquad \text{as } h\downarrow 0. \tag{9}$$

*If $0 < \widetilde{\gamma}^* \leqslant \widetilde{\gamma}_* < +\infty$, then $\widehat{\gamma}^* = \gamma^*$, $\widehat{\gamma}_* = \gamma_*$.*

**Proof.** The proof of the lemma repeats the outlines of the proof of limits of the logarithmic ratios (see Annex A.4 in [11]). For completeness we give this proof in Appendix. ■

Assume that for some $\gamma \in (0,1)$ the second order stochastic process $X$ satisfies conditions:

(C1) $\sigma_X(0,\delta) = \mathcal{O}(\delta^\gamma)$, as $\delta \downarrow 0$;

(C2) there exist a constant $\kappa > 0$ such that

$$\Lambda(\delta) := \sup_{\varphi(\delta)\leqslant t\leqslant T-\delta} \sup_{0<h\leqslant\delta} \left| \frac{\sigma_X(t,t+h)}{\kappa h^\gamma} - 1 \right| \longrightarrow 0 \qquad \text{as } \delta\downarrow 0$$

for every function $\varphi \in \Psi$.

For $(s,t) \in [0,T]^2$ set

$$c^2(s,t) := \frac{\sigma_X^2(s,t)}{\kappa^2 |t-s|^{2\gamma}} - 1. \tag{10}$$

It follows from $(C1)$ and $(C2)$ that for any $\varphi \in \Psi$

$$\begin{aligned}
\sup_{0\leqslant s\leqslant T-h} \sigma_X^2(s,s+h) \leqslant& \sup_{0\leqslant s\leqslant \varphi(h)} \sigma_X^2(s,s+h) + \sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma^2(s,s+h) \\
\leqslant& 4 \sup_{0\leqslant \delta\leqslant \varphi(h)+h} \sigma_X^2(0,\delta) + \kappa^2 h^{2\gamma} \Big( \sup_{\varphi(h)\leqslant s\leqslant T-h} |c^2(s,s+h)| + 1 \Big) \\
\leqslant& \mathcal{O}\big((\varphi(h))^{2\gamma}\big) + \kappa^2 h^{2\gamma} \left[\Lambda^2(h) + 2\Lambda(h) + 1\right] \longrightarrow 0 \quad \text{as } h\downarrow 0.
\end{aligned} \tag{11}$$

Thus the process $X$ is continuous in quadratic mean for all $s \in [0, T-h]$.

**Theorem 5** *Assume that for some constant $\gamma \in (0,1)$ the second order stochastic process $X$ satisfies conditions $(C1)$ and $(C2)$. Then the Orey index is equal to $\gamma_X$.*

**Proof.** By Lemma 4 it suffice to show that $\widehat{\gamma}_* = \widehat{\gamma}^* = \gamma_X$ and $\overline{\gamma}_* = \overline{\gamma}^* = \gamma_X$.

For simplicity, we shall omit index $X$ for $\gamma$. Observe first that condition $(C1)$ implies $\overline{\gamma}_* = \overline{\gamma}^* = \gamma$. Really,

$$\frac{\ln \sigma_X(0,h)}{\ln h} = \gamma + \frac{\ln(\mathcal{O}(h^\gamma)/h^\gamma)}{\ln h} \longrightarrow \gamma \qquad \text{as } h\downarrow 0.$$

It remains to prove $\widehat{\gamma}^* = \widehat{\gamma}_*$. By conditions $(C1)$ and $(C2)$ it follows that there exists $\delta_0$ such that for $\delta \leqslant \delta_0 < 1$ inequalities $\sigma_X(s,s+\delta) \leqslant 1/2$ and $\Lambda(\delta) < 1/2$ holds for all $0 \leqslant s \leqslant T-\delta_0$. Suppose that these inequalities are fulfill in the course of the proof of this theorem.

For $(s,t) \in [0,T]^2$ set

$$b(s,t) := \frac{\sigma_X(s,t)}{\kappa |t-s|^\gamma} - 1.$$

Assume that $-1/2 < b(s_0, s_0+\delta_0) \leqslant 0$ for some fixed $s_0 \in [\varphi(\delta_0), T-\delta_0]$. Furthermore, it is known that $-2x \leqslant \ln(1-x) \leqslant -x$ for $0 \leqslant x \leqslant 1/2$. Then by inequality above we get

$$\begin{aligned}\ln \sigma_X(s_0, s_0+\delta_0) &= \ln(\kappa\delta_0^\gamma) + \ln(1+b(s_0,s_0+\delta_0)) = \ln(\kappa\delta_0^\gamma) + \ln(1-(-b(s_0,s_0+\delta_0)))\\ &\leqslant \ln(\kappa\delta_0^\gamma) + b(s_0,s_0+\delta_0) \leqslant \ln(\kappa\delta_0^\gamma) + \Lambda(\delta_0)\end{aligned}$$

and

$$\begin{aligned}\ln \sigma_X(s_0, s_0+\delta_0) &\geqslant \ln(\kappa\delta_0^\gamma) + 2b(s_0,s_0+\delta_0) = \ln(\kappa\delta_0^\gamma) - 2|b(s_0,s_0+\delta_0)|\\ &\geqslant \ln(\kappa\delta_0^\gamma) - 2\Lambda(\delta_0)\end{aligned}$$

for any $\varphi \in \Psi$.

It is known that $|\ln(1+x)| \leqslant x$ for $x \geqslant 0$. Assume that $0 \leqslant b(s_0, s_0+\delta_0) < 1/2$ for some fixed $s_0 \in [\varphi(\delta_0), T-\delta_0]$, then

$$\begin{aligned}\ln \sigma_X(s_0, s_0+\delta_0) &= \ln(\kappa\delta_0^\gamma) + \ln(1+b(s_0,s_0+\delta_0)) \leqslant \ln(\kappa\delta_0^\gamma) + b(s_0,s_0+\delta_0)\\ &\leqslant \ln(\kappa\delta_0^\gamma) + \Lambda(\delta_0)\end{aligned}$$

and

$$\begin{aligned}\ln \sigma_X(s_0, s_0+\delta_0) &= \ln(\kappa\delta_0^\gamma) + \ln(1+b(s_0,s_0+\delta_0)) \geqslant \ln(\kappa\delta_0^\gamma) - 2|b(s_0,s_0+\delta_0)|\\ &\geqslant \ln(\kappa d_0^\gamma) - 2\Lambda(\delta_0)\end{aligned}$$

for any $\varphi \in \Psi$. Thus for every $s \in [\varphi(\delta_0), T-\delta_0]$ we obtain

$$\ln(\kappa\delta_0^\gamma) - 2\Lambda(\delta_0) \leqslant \ln \sigma_X(s, s+\delta_0) \leqslant \ln(\kappa\delta_0^\gamma) + \Lambda(\delta_0).$$

Consequently,

$$\begin{aligned}\gamma + \frac{\ln\kappa}{\ln\delta_0} - \frac{\Lambda(\delta_0)}{|\ln\delta_0|} &\leqslant \inf_{\varphi(\delta_0)\leqslant s\leqslant T-\delta_0} \frac{\ln\sigma_X(s,s+\delta_0)}{\ln\delta_0} \leqslant \sup_{\varphi(\delta_0)\leqslant s\leqslant T-\delta_0} \frac{\ln\sigma_X(s,s+\delta_0)}{\ln\delta_0}\\ &\leqslant \gamma + \frac{\ln\kappa}{\ln\delta_0} + 2\,\frac{\Lambda(\delta_0)}{|\ln\delta_0|}\,.\end{aligned}$$

and both sides of the above inequality goes to $\gamma$ as $\delta_0 \to 0$. Thus $\widehat{\gamma}_* = \widehat{\gamma}^* = \gamma_X$. ■

## 2.1 Subfractional Brownian motion

We shall prove that sfBm satisfies conditions $(C1)$ and $(C2)$.

**Definition 6** ([1]) *A* ***sub-fractional Brownian motion*** *with index $H$, $H \in (0,1)$, is a mean zero Gaussian stochastic process $S^H = (S_t^H, t \geqslant 0)$ with covariance function*

$$G_H(s,t) := s^{2H} + t^{2H} - \frac{1}{2}\big[(s+t)^{2H} + |s-t|^{2H}\big].$$

The incremental variance function of sfBm is of the following form

$$\sigma^2_{S^H}(s,t) = \mathbf{E}|S_t^H - S_s^H|^2 = |t-s|^{2H} + (s+t)^{2H} - 2^{2H-1}(t^{2H} + s^{2H}). \tag{12}$$

Since for any $0 \leqslant s \leqslant t \leqslant T$ inequalities (see [1])

$$(t-s)^{2H} \leqslant \sigma^2_{S^H}(s,t) \leqslant (2 - 2^{2H-1})(t-s)^{2H}, \qquad \text{if} \quad 0 < H < 1/2, \tag{13}$$

$$(2 - 2^{2H-1})(t-s)^{2H} \leqslant \sigma^2_{S^H}(s,t) \leqslant (t-s)^{2H}, \qquad \text{if} \quad 1/2 < H < 1 \tag{14}$$

holds, then condition $(C1)$ is satisfied.

From incremental variance function (12) we get

$$\sigma^2_{S^H}(s, s+h) = h^{2H} + f_s(h),$$

where

$$f_s(h) := (2s+h)^{2H} - 2^{2H-1}\big[s^{2H} + (s+h)^{2H}\big].$$

Note that

$$f_s(0) = f_s'(0) = 0.$$

By Taylor formula we obtain

$$
\begin{aligned}
f_s(h) =& f_s(0) + f_s'(0)h + \int_0^h f_s''(x)(h-x)\,dx = \int_0^h f_s''(x)(h-x)\,dx \\
=& 2H(2H-1)\int_0^h \left[(2s+x)^{2H-2} - 2^{2H-1}(s+x)^{2H-2}\right](h-x)\,dx.
\end{aligned}
$$

From inequality

$$
\begin{aligned}
&\left[2^{2H-1}(s+x)^{2H-2} - (2s+x)^{2H-2}\right] = \frac{1}{(s+x)^{2-2H}}\left[2^{2H-1} - \left(\frac{s+x}{2s+x}\right)^{2-2H}\right] \\
&= \frac{1}{(s+x)^{2-2H}}\left[2^{2H-1} - \left(1 - \frac{s}{2s+x}\right)^{2-2H}\right] \leqslant \frac{1}{(s+x)^{2-2H}}\left[2^{2H-1} - 2^{-1}\right],
\end{aligned}
$$

it follows that for $s > 0$

$$
|f_s(h)| \leqslant (2^{2H}-1)\int_0^h \frac{h-x}{(s+x)^{2-2H}}\,dx \leqslant \frac{1}{2}\,(2^{2H}-1)s^{2H-2}h^2
$$

and

$$
\begin{aligned}
\sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\left|\frac{\sigma^2_{SH}(s,s+h)}{h^{2H}} - 1\right| &= \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\frac{|f_s(h)|}{h^{2H}} \\
&\leqslant \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\frac{2^{2H-1}\delta^{2-2H}}{s^{2-2H}} \leqslant \frac{2^{2H-1}}{(L(\delta))^{2-2H}}
\end{aligned}
$$

for every $\varphi \in \Psi$, where $L(h) = \varphi(h)/h$. So we get condition $(C2)$ with $\kappa = 1$.

**Remark 7** *In condition $(C2)$ the function $\varphi(\delta)$ we could not change by $\delta$ or $0$. Really, let $H > 1/2$. Then*

$$
\begin{aligned}
&\sup_{0\leqslant s\leqslant T-\delta}\ \sup_{0\leqslant h\leqslant\delta}|h^{-2H}f_s(h)| \geqslant \sup_{\delta\leqslant s\leqslant T-\delta}\ \sup_{0\leqslant h\leqslant\delta}|h^{-2H}f_s(h)| \\
&= 2H(2H-1)\sup_{\delta\leqslant s\leqslant T-\delta}\ \sup_{0\leqslant h\leqslant\delta}\int_0^h\left[\frac{2^{2H-1}}{h^{2H}(s+x)^{2-2H}} - \frac{1}{h^{2H}(2s+x)^{2-2H}}\right](h-x)\,dx \\
&\geqslant 2H(2H-1)\sup_{\delta\leqslant s\leqslant T-\delta}\ \sup_{0\leqslant h\leqslant\delta}\int_0^h\frac{2^{2H-1}-1}{h^{2H}(2s+x)^{2-2H}}\,(h-x)\,dx \\
&\geqslant H(2H-1)\sup_{\delta\leqslant s\leqslant T-\delta}\ \sup_{0\leqslant h\leqslant\delta}\frac{(2^{2H-1}-1)h^{2-2H}}{(2s+h)^{2-2H}} \\
&= H(2H-1)(2^{2H-1}-1)\sup_{\delta\leqslant s\leqslant T-\delta}\frac{\delta^{2-2H}}{(2s+\delta)^{2-2H}} \\
&= H(2H-1)(2^{2H-1}-1)3^{2H-2}.
\end{aligned}
$$

## 2.2 Bifractional Brownian motion

**Definition 8** ([7]) *A **bifractional Brownian motion** $B^{HK} = (B_t^{HK}, t \geqslant 0)$ with parameters $H \in (0,1)$ and $K \in (0,1]$ is a centered Gaussian process with covariance function*

$$
R_{HK}(t,s) = 2^{-K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right), \qquad s,t \geqslant 0.
$$

The incremental variance function of bifBm is of the following form

$$
\sigma^2_{B^{H,K}}(s,t) = \mathbf{E}|B_t^{H,K} - B_s^{H,K}|^2 = 2^{1-K}\left[|t-s|^{2HK} - (t^{2H}+s^{2H})^K\right] + t^{2HK} + s^{2HK}.
$$

Let $H \in (0,1)$ and $K \in (0,1]$. Then

$$
2^{-K}|t-s|^{2HK} \leqslant \sigma^2_{B^{H,K}}(s,t) \leqslant 2^{1-K}|t-s|^{2HK} \tag{15}
$$

for all $s,t \in [0,\infty)$ (see [7]). Thus condition $(C1)$ holds.

Since

$$
\sigma^2_{B^{H,K}}(s,s+h) = 2^{1-K}(h^{2HK} - f_s(h))
$$

with

$$f_s(h) := \left[s^{2H} + (s+h)^{2H}\right]^K - 2^{K-1}\left[s^{2HK} + (s+h)^{2HK}\right],$$

then $f_s(0) = f_s'(0) = 0$ and by Taylor formula we obtain

$$\frac{\sigma^2_{B^{HK}}(s,s+h)}{2^{1-K}h^{2HK}} - 1 = -h^{-2HK}\int_0^h f_s''(x)(h-x)\,dx,$$

where

$$\begin{aligned} f_s''(x) =& 4K(K-1)H^2\left[s^{2H} + (s+x)^{2H}\right]^{K-2}(s+x)^{2(2H-1)} \\ &+ 2HK(2H-1)\left[s^{2H} + (s+x)^{2H}\right]^{K-1}(s+x)^{2H-2} \\ &- 2^K HK(2HK-1)(s+x)^{2HK-2}. \end{aligned}$$

Note that for $H \geqslant 1/2$

$$\frac{(s+x)^{2(2H-1)}}{[s^{2H} + (s+x)^{2H}]^{2-K}} = \left[\frac{(s+x)^{2H}}{s^{2H} + (s+x)^{2H}}\right]^{2-K}(s+x)^{2HK-2} \leqslant (s+x)^{2HK-2}.$$

Thus for $s > 0$

$$\begin{aligned} \sup_{0\leqslant x\leqslant h}|f_s''(x)| \leqslant& \frac{4}{s^{2-2HK}}\mathbf{1}_{\{H\geqslant 1/2\}} + \frac{4}{(2s^{2H})^{2-K}s^{2(1-2H)}}\mathbf{1}_{\{H<1/2\}} \\ &+ \frac{2}{(2s^{2H})^{1-K}s^{2-2H}} + \frac{2}{s^{2-2HK}} \leqslant \frac{8}{s^{2-2HK}} \end{aligned}$$

and

$$\sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\left|\frac{\sigma^2_{B_{HK}}(s,s+h)}{2^{1-K}h^{2H}} - 1\right| \leqslant \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\frac{8\delta^{2-2HK}}{s^{2-2HK}} \leqslant \frac{8}{(L(\delta))^{2-2HK}}$$

for every $\varphi \in \Psi$. So condition $(C2)$ holds.

## 2.3 Ornstein-Uhlenbeck process

The fractional Ornstein-Uhlenbeck (fO-U) process of the first kind is the unique solution of the following stochastic differential equation

$$X_t = x_0 - \mu\int_0^t X_s\,ds + \theta B_t^H, \qquad t \leqslant T, \tag{16}$$

with $\mu, \theta > 0$, where $B^H$, $0 < H < 1$, is a fBm. It has explicit solution

$$X_t = x_0 e^{-\mu t} + \theta\int_0^t e^{-\mu(t-u)}dB_u^H,$$

where the integral exists as a Riemann-Stieltjes integral for all $t > 0$ (see, e.g., [4]).

First of all we verify condition $(C1)$. From [4] we know that

$$\int_0^t e^{\mu u}dB_u^H = e^{\mu t}B_t^H - \mu\int_0^t e^{\mu u}B_u^H du.$$

Thus

$$X_t^2 \leqslant 2x_0^2 + 2\theta^2\left(\int_0^t e^{\mu u}dB_u^H\right)^2 \leqslant 2x_0^2 + 4\theta^2\left(e^{2\mu t}(B_t^H)^2 + \mu^2 t\int_0^t e^{2\mu u}(B_u^H)^2 du\right)$$

and

$$\sup_{t\leqslant T}\mathbf{E}X_t^2 \leqslant 2x_0^2 + 4\theta^2 e^{2\mu T}T^{2H}(1 + \mu^2 T^2).$$

From (16) we get

$$\sigma_X^2(0,h) \leqslant 2\mu^2\mathbf{E}\left(\int_0^h X_t\,dt\right)^2 + 2\theta^2\mathbf{E}(B_h^H)^2 \leqslant 2\mu^2 h^2\sup_{t\leqslant h}\mathbf{E}X_t^2 + 2\theta^2 h^{2H} \leqslant Ch^{2H}.$$

This proves a condition $(C1)$.

The incremental variance function of $X$ has the following form

$$\sigma_X^2(t,t+h) = \mu^2 \mathbf{E}\Big(\int_t^{t+h} X_s\,ds\Big)^2 - 2\mu\theta \mathbf{E}\Big([B^H(t+h)-B^H(t)]\int_t^{t+h} X_s\,ds\Big) + \theta^2 \sigma_{B^H}^2(t,t+h).$$

Cauchy-Schwarz inequality yields

$$\mathbf{E}\Big([B^H(t+h)-B^H(t)]\int_t^{t+h} X_s\,ds\Big) \leqslant \mathbf{E}^{1/2}[B^H(t+h)-B^H(t)]^2\Big(h\int_t^{t+h}\mathbf{E}X_s^2\,ds\Big)^{1/2} \leqslant h^{H+1}\Big(\sup_{t\leqslant s\leqslant t+h}\mathbf{E}X_s^2\Big)^{1/2}.$$

Thus for every $\varphi \in \Psi$

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\left|\frac{\sigma_X^2(t,t+h)}{\theta^2 h^{2H}}-1\right| \leqslant \theta^{-2}\delta^{1-H}\Big[\delta^{1-H}\mu^2\sup_{t\leqslant T}\mathbf{E}X_t^2 + 2\mu\theta\Big(\sup_{t\leqslant T}\mathbf{E}X_t^2\Big)^{1/2}\Big] \longrightarrow 0$$

as $\delta \downarrow 0$. Condition $(C2)$ follow from the inequality

$$\left|\frac{\sigma_X(t,t+h)}{\theta h^H}-1\right| = \left|\frac{\sigma_X^2(t,t+h)}{\theta^2 h^{2H}}-1\right| \Big/ \left|\frac{\sigma_X(t,t+h)}{\theta h^H}+1\right| \leqslant \left|\frac{\sigma_X^2(t,t+h)}{\theta^2 h^{2H}}-1\right|.$$

## 2.4 Fractional Brownian bridge

The fractional Brownian bridge is defined in $[0,T]$ by

$$X_t^H = B_t^H - \frac{t^{2H}+T^{2H}-|t-T|^{2H}}{2T^{2H}}\,B_T^H.$$

where $B^H$, $0<H<1$, is a fBm in $[0,T]$.

Now we verify condition $(C1)$. The incremental variance function of $X^H$ has the following form

$$\sigma_{X^H}^2(t,t+h) = h^{2H} - \frac{1}{4T^{2H}}\,f_t^2(h)$$

where

$$f_t^2(h) := \big[(t+h)^{2H}-t^{2H}-|t+h-T|^{2H}+|t-T|^{2H}\big]^2.$$

Thus

$$\sigma_{X^H}^2(t,t+h) \leqslant h^{2H}. \tag{17}$$

So condition $(C1)$ is satisfied.

Assume $H<1/2$. Since

$$\big|(t+h)^{2H}-t^{2H}\big| \leqslant h^{2H} \quad\text{and}\quad \big|(T-t-h)^{2H}-(T-t)^{2H}\big| \leqslant h^{2H}$$

then for every $\varphi\in\Psi$

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\left|\frac{\sigma_{X^H}^2(t,t+h)}{h^{2H}}-1\right| = \frac{1}{4T^{2H}}\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\frac{f_t^2(h)}{h^{2H}} \leqslant T^{-2H}\delta^{2H}.$$

Assume $H\geqslant 1/2$. Then $f_t(0)=0$ and by Taylor formula we obtain

$$\frac{\sigma_{X^H}^2(t,t+h)}{h^{2H}} - 1 = -\frac{1}{4T^{2H}h^{2H}}\Big(\int_0^h f_t'(x)\,dx\Big)^2,$$

where

$$f_t'(x) = 2H\big[(t+x)^{2H-1}-(T-t-x)^{2H-1}\big].$$

Thus for every $\varphi\in\Psi$ and $H\geqslant 1/2$ we get

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\ \sup_{0<h\leqslant\delta}\left|\frac{\sigma_{X^H}^2(t,t+h)}{h^{2H}}-1\right| \leqslant \frac{\delta^{2-2H}}{4T^{2H}}\cdot 4H^2T^{4H-2} = H^2T^{2H-2}\delta^{2-2H}.$$

# 3 The convergence of the second order quadratic variation of process $X$ along irregular partition

Let $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $T > 0$, be a sequence of partitions of the interval $[0,T]$ and $(N_n)$ is an increasing sequence of natural numbers. Such sequence of partitions is called irregular. Define

$$m_n = \max_{1\leqslant k\leqslant N_n} \Delta_k^n t, \qquad p_n = \min_{1\leqslant k\leqslant N_n} \Delta_k^n t, \qquad \Delta_k^n t = t_k^n - t_{k-1}^n .$$

Usually in practice observations of the process are available at discrete regular time intervals. However, it may happen that part of the observations are lost, resulting in observations at irregular time intervals.

**Definition 9** *A sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ is regular if we have $m_n = p_n = TN_n^{-1}$ for all $n \in \mathbb{N}$ or, equivalently, $t_k^n = \frac{kT}{N_n}$ for all $n \in \mathbb{N}$ and all $k \in \{0, \ldots, N_n\}$.*

**Definition 10** *The second order quadratic variations of Gaussian processes $X$ along the partitions $(\pi_n)_{n\in\mathbb{N}}$ with Orey index $\gamma$ is defined by*

$$V_{\pi_n}^{(2)}(X,2) = 2\sum_{k=1}^{N_n-1} \frac{\Delta_{k+1}^n t(\Delta_{ir,k}^{(2)n} X)^2}{(\Delta_k^n t)^{\gamma+1/2}(\Delta_{k+1}^n t)^{\gamma+1/2}[\Delta_k^n t + \Delta_{k+1}^n t]},$$

*where*

$$\Delta_{ir,k}^{(2)n} X_k^n = \Delta_k^n t X(t_{k+1}^n) + \Delta_{k+1}^n t X(t_{k-1}^n) - (\Delta_k^n t + \Delta_{k+1}^n t) X(t_k^n).$$

If the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular then one has

$$V_{N_n}^{(2)}(X,2) = (T^{-1}N_n)^{2\gamma-1} \sum_{k=1}^{N_n-1} \left(\Delta_{n,k}^{(2)} X\right)^2, \qquad \Delta_{n,k}^{(2)} X = X(t_{k+1}^n) - 2X(t_k^n) + X(t_{k-1}^n).$$

To study the almost sure convergence of the second order quadratic variation of $X$ we need additional assumptions on the sequence $(\pi_n)_{n\in\mathbb{N}}$.

**Definition 11** (see [2]) *Let $(\ell_k)_{k\geqslant 1}$ be a sequence of real numbers in the interval $(0,\infty)$. We say that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of partitions with asymptotic ratios $(\ell_k)_{k\geqslant 1}$ if it satisfies the following assumptions:*

1. *There exists $c \geqslant 1$ such that $m_n \leqslant cp_n$ for all $n$.*
2. $\lim_{n\to\infty} \sup_{1\leqslant k\leqslant N_n} \left| \frac{\Delta_{k-1,n} t}{\Delta_k^n t} - \ell_k \right| = 0.$

*The set $\mathcal{L} = \{\ell_1; \ell_2; \ldots; \ell_k; \ldots\}$ will be called the range of the asymptotic ratios of the sequence $(\pi_n)_{n\in\mathbb{N}}$.*

It is clear that if the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular, then it is a sequence with asymptotic ratios $\ell_k = 1$ for all $k \geqslant 1$.

**Definition 12** (see [2]) *The function $g : (0,\infty) \to \mathbb{R}$ is invariant on $\mathcal{L}$ if for all $\ell, \hat{\ell} \in \mathcal{L}$, $g(\ell) = g(\hat{\ell})$.*

For example, let $\mathcal{L} = \{\alpha, \alpha^{-1}\}$ be the set containing two real positive numbers and let

$$g(\lambda) = \frac{1 + \lambda^{2\gamma-1} - (1+\lambda)^{2\gamma-1}}{\lambda^{\gamma-1/2}}.$$

The function $g$ is invariant on $\mathcal{L}$.

**Proposition 13** *Let $X = \{X(t) : t \in [0,T]\}$, $T > 0$, be a mean zero second order process satisfying conditions $(C1)$ and $(C2)$. Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions with asymptotic ratios $(\ell_k)_{k\geqslant 1}$ and range of the asymptotic ratios $\mathcal{L}$. If the function $g$ is invariant on $\mathcal{L}$ or the sequence of functions $\ell_n(t)$ converges uniformly to $\ell(t)$ on the interval $[0,T]$, where*

$$\ell_n(t) = \sum_{k=1}^{N_n-1} \ell_k \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t),$$

*then*

$$\lim_{n\to\infty} \mathbf{E} V^{(2)}_{\pi_n}(X,2) = 2\kappa^2 \int_0^T g(\ell(t))\,dt,$$

*where*

$$g(\lambda) = \frac{1+\lambda^{2\gamma-1} - (1+\lambda)^{2\gamma-1}}{\lambda^{\gamma-1/2}}\,.$$

**Proof.** Rewrite the expectation of the increments of the second order irregular variation in the following way

$$\begin{aligned}
\mathbf{E}(\Delta^{(2)n}_{ir,k}X)^2 =&(\Delta^n_k t)^2\sigma^2_X(t^n_k,t^n_{k+1}) + (\Delta^n_{k+1}t)^2\sigma^2_X(t^n_{k-1},t^n_k)\\
&+\Delta^n_k t\cdot\Delta^n_{k+1}t\big[\sigma^2_X(t^n_k,t^n_{k+1}) - \sigma^2_X(t^n_{k-1},t^n_{k+1}) + \sigma^2_X(t^n_{k-1},t^n_k)\big]\\
=&[\Delta^n_k t+\Delta^n_{k+1}t]\big[\Delta^n_k t\cdot\sigma^2_X(t^n_k,t^n_k+\Delta^n_{k+1}t) + \Delta^n_{k+1}t\cdot\sigma^2_X(t^n_{k-1},t^n_{k-1}+\Delta^n_k t)\big]\\
&-\Delta^n_k t\cdot\Delta^n_{k+1}t\cdot\sigma^2_X(t^n_{k-1},t^n_{k-1}+\Delta^n_k t+\Delta^n_{k+1}t)\\
=&I^{(1)}_k - I^{(2)}_k + I^{(3)}_k,
\end{aligned}$$

where

$$\begin{aligned}
I^{(1)}_k :=&[\Delta^n_k t+\Delta^n_{k+1}t]\big\{\Delta^n_k t\big[\sigma^2_X(t^n_k,t^n_{k+1}) - \kappa^2(\Delta^n_{k+1}t)^{2\gamma}\big]\\
&+\Delta^n_{k+1}t\big[\sigma^2_X(t^n_{k-1},t^n_k) - \kappa^2(\Delta^n_k t)^{2\gamma}\big]\big\},\\
I^{(2)}_k :=&\Delta^n_k t\cdot\Delta^n_{k+1}t\big[\sigma^2_X(t^n_{k-1},t^n_{k+1}) - \kappa^2(\Delta^n_k t+\Delta^n_{k+1}t)^{2\gamma}\big],\\
I^{(3)}_k :=&\kappa^2[\Delta^n_k t+\Delta^n_{k+1}t]\Delta^n_k t\cdot\Delta^n_{k+1}t\big\{(\Delta^n_{k+1}t)^{2\gamma-1} + (\Delta^n_k t)^{2\gamma-1} - (\Delta^n_k t+\Delta^n_{k+1}t)^{2\gamma-1}\big\}.
\end{aligned}$$

Set

$$\mu^n_k = [\Delta^n_k t+\Delta^n_{k+1}t](\Delta^n_{k+1}t)^{\gamma+1/2}(\Delta^n_k t)^{\gamma+1/2} \quad\text{and}\quad \ell^n_k = \frac{\Delta^n_k t}{\Delta^n_{k+1}t}\,.$$

Then

$$\begin{aligned}
I^{(1)}_k =&\kappa^2[\Delta^n_k t+\Delta^n_{k+1}t]\Delta^n_k t\cdot\Delta^n_{k+1}t\big[(\Delta^n_{k+1}t)^{2\gamma-1}c^2(t^n_k,t^n_{k+1}) + (\Delta^n_k t)^{2\gamma-1}c^2(t^n_{k-1},t^n_k)\big]\\
=&\kappa^2\mu^n_k\big[(\ell^n_k)^{1/2-\gamma}c^2(t^n_k,t^n_{k+1}) + (\ell^n_k)^{\gamma-1/2}c^2(t^n_{k-1},t^n_k)\big],\\
I^{(2)}_k =&\kappa^2\mu^n_k(\Delta^n_k t)^{1/2-\gamma}(\Delta^n_{k+1}t)^{1/2-\gamma}(\Delta^n_k t+\Delta^n_{k+1}t)^{2\gamma-1}c^2(t^n_{k-1},t^n_{k+1})\\
=&\kappa^2\mu^n_k(\ell^n_k)^{1/2-\gamma}(1+\ell^n_k)^{2\gamma-1}c^2(t^n_{k-1},t^n_{k+1})
\end{aligned}$$

and

$$\begin{aligned}
I^{(3)}_k =&\kappa^2\mu^n_k\big((\ell^n_k)^{1/2-\gamma} + (\ell^n_k)^{\gamma-1/2} - (\ell^n_k)^{1/2-\gamma}(1+\ell^n_k)^{2\gamma-1}\big)\\
=&\kappa^2\mu^n_k(\ell^n_k)^{1/2-\gamma}\big[1+(\ell^n_k)^{2\gamma-1} - (1+\ell^n_k)^{2\gamma-1}\big],
\end{aligned}$$

where the function $c^2(s,t)$ is defined in (10). We further observe that

$$\begin{aligned}
\mathbf{E}V^{(2)}_{\pi_n}(X,2) =&2\sum_{k=1}^{\tau_n+1}\frac{\Delta^n_{k+1}t\cdot\mathbf{E}(\Delta^{(2)n}_{ir,k}X)^2}{\mu^n_k} + 2\sum_{k=\tau_n+2}^{N_n-1}\frac{\Delta^n_{k+1}t\cdot\mathbf{E}(\Delta^{(2)n}_{ir,k}X)^2}{\mu^n_k}\\
=&2\sum_{k=1}^{\tau_n+1}\frac{\Delta^n_{k+1}t\cdot\mathbf{E}(\Delta^{(2)n}_{ir,k}X)^2}{\mu^n_k} + 2\kappa^2\sum_{k=\tau_n+2}^{N_n-1}\Delta^n_{k+1}t\cdot J^{(1)}_k\\
&+2\kappa^2\sum_{k=\tau_n+2}^{N_n-1}\Delta^n_{k+1}t\cdot J^{(2)}_k,
\end{aligned}\tag{18}$$

where $\tau_n = [\varphi(m_n)N_n]$, $[a]$ is an integer part of a real number $a$,

$$\begin{aligned}
J^{(1)}_k =&(\ell^n_k)^{1/2-\gamma}\big[c^2(t^n_k,t^n_{k+1}) + \ell^{2\gamma-1}c^2(t^n_{k-1},t^n_k) - (1+\ell^n_k)^{2\gamma-1}c^2(t^n_{k-1},t^n_{k+1})\big],\\
J^{(2)}_k =&(\ell^n_k)^{1/2-\gamma}\big[1+\ell^{2\gamma-1}_k - (1+\ell^n_k)^{2\gamma-1}\big].
\end{aligned}$$

Now we estimate the first term of equality (18). Note that

$$\tau_n \leqslant \frac{\varphi(m_n)}{p_n} \leqslant cL(m_n), \qquad 2p_n^{2\gamma+2} \leqslant \mu_k^n \leqslant 2m_n^{2\gamma+2}, \tag{19}$$

$$\sum_{k=1}^{\tau_n+1} \Delta t_{k+1}^n \leqslant \varphi(m_n)\frac{m_n}{p_n} + m_n \leqslant 2c\varphi(m_n). \tag{20}$$

By conditions $(C1)$, $(C2)$, and inequalities (19), (20) we get

$$\begin{aligned}
&2\sum_{k=1}^{\tau_n+1} \frac{\Delta_{k+1}^n t \cdot \mathbf{E}(\Delta_{ir,k}^{(2)n} X)^2}{\mu_k^n} \\
&\leqslant \frac{8c^3\varphi(m_n)}{p_n^{2\gamma}} \max_{1\leqslant k\leqslant \tau_n+2} \sigma_X^2(t_{k-1}^n, t_k^n) \leqslant \frac{32c^3\varphi(m_n)}{p_n^{2\gamma}} \sup_{1\leqslant k\leqslant \tau_n+2} \sigma_X^2(0, t_k^n) \\
&\leqslant \frac{32c^3\varphi(m_n)}{p_n^{2\gamma}} \max_{1\leqslant k\leqslant \tau_n+2} \mathcal{O}\big((t_k^n)^{2\gamma}\big) = \frac{32c^3\varphi(m_n)}{p_n^{2\gamma}} O\big(\big((cL(m_n)+2)m_n\big)^{2\gamma}\big) \\
&\leqslant 32c^3\varphi(m_n)\, O\big(\big(cL(m_n)+2\big)^{2\gamma}\big)
\end{aligned}$$

as $m_n \downarrow 0$. From the properties of function $\varphi$ we obtain that the right hand side of the above inequality tends to zero as $m_n \downarrow 0$.

Next, since $[\varphi(m_n)N_n] + 1 \geqslant \varphi(m_n)$, for the second term of equality (18) we get

$$\begin{aligned}
&2\kappa^2 \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot J_k^{(1)} \\
&\leqslant 2\kappa^2 \max_{\tau_n+1\leqslant k\leqslant N_n-1} \big|c^2(t_k^n, t_{k+1}^n)\big| \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t\big[(\ell_k^n)^{1/2-\gamma} + (\ell_k^n)^{\gamma-1/2}\big] \\
&\quad + 2\kappa^2 \max_{\tau_n+2\leqslant k\leqslant N_n-1} \big|c^2(t_{k-1}^n, t_{k+1}^n)\big| \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot (\ell_k^n)^{1/2-\gamma}(1+\ell_k^n)^{2\gamma-1} \\
&\leqslant 2\kappa^2 T \sup_{\varphi(m_n)\leqslant t\leqslant T-m_n} \sup_{0<h\leqslant m_n} \big|c^2(t, t+h)\big| \max_{1\leqslant k\leqslant N_n} \big[(\ell_k^n)^{1/2-\gamma} + (\ell_k^n)^{\gamma-1/2}\big] \\
&\quad + 2\kappa^2 T \sup_{\varphi(m_n)\leqslant t\leqslant T-2m_n} \sup_{0<h\leqslant m_n} \big|c^2(t, t+2h)\big| \max_{1\leqslant k\leqslant N_n} \big[(\ell_k^n)^{1/2-\gamma}(1+\ell_k^n)^{2\gamma-1}\big] \\
&\leqslant 2\kappa^2 T\big[\Lambda^2(m_n) + 2\Lambda(m_n)\big] \max_{1\leqslant k\leqslant N_n} \big[(\ell_k^n)^{1/2-\gamma} + (\ell_k^n)^{\gamma-1/2}\big] \\
&\quad + 2\kappa^2 T\big[\Lambda^2(2m_n) + 2\Lambda(2m_n)\big] \max_{1\leqslant k\leqslant N_n} \big[(\ell_k^n)^{1/2-\gamma}(1+\ell_k^n)^{2\gamma-1}\big] \\
&\leqslant 4\kappa^2 Tc\big[\Lambda^2(m_n) + 2\Lambda(m_n)\big] + 2\kappa^2 T(1+c)c\big[\Lambda^2(2m_n) + 2\Lambda(2m_n)\big].
\end{aligned}$$

Thus the second term of equality (18) tends to zero as $n \to \infty$.

It still remains to investigate asymptotic behavior of the third term of equality (18). If the function $g$ is invariant on $\mathcal{L}$, then

$$\begin{aligned}
2\kappa^2 \sum_{k=\tau_n+1}^{N_n-1} \Delta_{k+1}^n t \cdot J_k^{(2)} &= 2\kappa^2 \sum_{k=\tau_n+1}^{N_n-1} \Delta_{k+1}^n t\, \frac{1 + \ell_k^{2\gamma-1} - (1+\ell_k^n)^{2\gamma-1}}{(\ell_k^n)^{\gamma-1/2}} \\
&= 2\kappa^2 g(\ell)T - 2\kappa^2 g(\ell) \sum_{k=1}^{\tau_n} \Delta_{k+1}^n t \longrightarrow 2\kappa^2 g(\ell)T \quad \text{as } n \to \infty
\end{aligned}$$

for all $\ell \in \mathcal{L}$ by the inequality (20). If the sequence of functions $\ell_n(t)$ converges uniformly to $\ell(t)$ on the interval $[0, T]$, then

$$\begin{aligned}
2\kappa^2 \sum_{k=\tau_n+1}^{N_n-1} \Delta_{k+1}^n t \cdot J_k^{(2)} &= 2\kappa^2 \int_0^T g(\ell_n(t))\, dt - \sum_{k=1}^{\tau_n} \Delta_{k+1}^n t \cdot J_k^{(2)} \\
&\longrightarrow 2\kappa^2 \int_0^T g(\ell(t))\, dt
\end{aligned}$$

since

$$\sum_{k=1}^{\tau_n} \Delta_{k+1}^n t \cdot J_k^{(2)} \leqslant 2(1+c)c^{1/2}\varphi(m_n) \longrightarrow 0 \quad \text{as } n\to\infty.$$

Thus

$$\mathbf{E}V_{\pi_n}^{(2)}(X,2) \longrightarrow 2\kappa^2 \int_0^T g(\ell(t))\,dt \qquad \text{as } n\to\infty.$$

■

**Corollary 14** *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval $[0,T]$, $T>0$, and let $X=\{X(t): t\in[0,T]\}$, $T>0$, be a mean zero second order process satisfying conditions $(C1)$ and $(C2)$. Then*

$$\mathbf{E}V_{N_n}^{(2)}(X,2) \longrightarrow \kappa^2(4-2^{2\gamma})T \qquad \text{as } n\to\infty.$$

**Proof.** For regular subdivision $\ell_k=1$. Thus $g(\lambda)=2-2^{2\gamma-1}$ and the statement of the corollary follows immediately from Proposition 13. ■

Now we give a little more general version of the statement of Corollary 14.

**Proposition 15** *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval $[0,T]$, $T>0$. Assume that condition $(C1)$ is fulfilled for some constant $\gamma\in(0,1)$ and there exists a continuous bounded function $g_0:(0,T)\to\mathbb{R}$ such that*

$$\lim_{h\to 0+}\sup_{\varphi(h)\leqslant t\leqslant T-h}\left|\frac{\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2}{h^{2\gamma}}-g_0(t)\right|=0. \tag{21}$$

*Then*

$$\mathbf{E}V_{N_n}^{(2)}(X,2)\longrightarrow\int_0^T g_0(t)\,dt \quad \text{as } n\to\infty.$$

**Proof.** Note that

$$\begin{aligned}
&\left|\mathbf{E}V_{N_n}^{(2)}(X,2)-\int_0^T g_0(t)\,dt\right|\\
&\quad\leqslant\left(\frac{T}{N_n}\right)^{1-2\gamma}\sum_{k=1}^{\tau_n}\mathbf{E}(\Delta_{n,k}^{(2)}X)^2+\frac{T}{N_n}\sum_{k=\tau_n+1}^{N_n-1}\left|\frac{\mathbf{E}(\Delta_{n,k}^{(2)}X)^2}{T^{2\gamma}N_n^{-2\gamma}}-g_0\Big(\frac{kT}{N_n}\Big)\right|\\
&\qquad+\left|\int_0^T g_0(t)\,dt-\frac{T}{N_n}\sum_{k=\tau_n+1}^{N_n-1}g_0\Big(\frac{kT}{N_n}\Big)\right|,
\end{aligned} \tag{22}$$

where $\tau_n=[\varphi(TN_n^{-1})N_n]$. By condition $(C1)$ we get

$$\max_{1\leqslant k\leqslant\tau_n+1}\sigma^2(t_{k-1}^n,t_k^n)\leqslant 2\sup_{1\leqslant k\leqslant\tau_n+1}\sigma^2(0,t_k^n)=\mathcal{O}\big((TN_n^{-1}(\tau_n+1))^{2\gamma}\big)=\mathcal{O}\big((\varphi(TN_n^{-1}))^{2\gamma}\big).$$

Thus

$$\begin{aligned}
\left(\frac{T}{N_n}\right)^{1-2\gamma}\sum_{k=1}^{\tau_n}\mathbf{E}(\Delta_{n,k}^{(2)}X)^2&\leqslant 4T\left(\frac{T}{N_n}\right)^{-2\gamma}\varphi\left(\frac{T}{N_n}\right)\max_{1\leqslant k\leqslant\tau_n+1}\sigma^2(t_{k-1}^n,t_k^n)\\
&=4T\left(\frac{T}{N_n}\right)^{-2\gamma}\varphi\left(\frac{T}{N_n}\right)\mathcal{O}\left(\left(\varphi\left(\frac{T}{N_n}\right)\right)^{2\gamma}\right)
\end{aligned}$$

and the first term in inequality (22) tends to zero as $n\to\infty$.

Assumption (21) yields

$$\begin{aligned}
&\max_{\tau_n+1\leqslant k\leqslant N_n-1}\left|\frac{\mathbf{E}(\Delta_{n,k}^{(2)}X)^2}{T^{2\gamma}N_n^{-2\gamma}}-g_0\Big(\frac{kT}{N_n}\Big)\right|\\
&\quad\leqslant\sup_{\varphi(TN_n^{-1})\leqslant t\leqslant T-TN_n^{-1}}\left|\frac{\mathbf{E}\big(X_{t+TN_n^{-1}}-2X_t+X_{t-TN_n^{-1}}\big)^2}{(TN_n^{-1})^{2\gamma}}-g_0(t)\right|\longrightarrow 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

The third term of the right hand side of (22) also converges towards 0 as $n \to \infty$ as a consequence of classical results for Riemann sums and inequality

$$\frac{T}{N_n}\sum_{k=1}^{\tau_n}\left|g_0\Big(\frac{kT}{N_n}\Big)\right| \leqslant \sup_{0\leqslant t\leqslant T}|g_0(t)|\,\varphi\Big(\frac{T}{N_n}\Big).$$

■

**Theorem 16** *Assume that conditions of Proposition 13 are satisfied and the partition $\pi_n$ is such that $p_n \overset{\mathbf{n}\to\infty}{=} o(\ln^{-1} n)$. Moreover assume that $X$ is a Gaussian process with the Orey index $\gamma$ and*

$$\max_{1\leqslant k\leqslant N_n-1}\sum_{j=1}^{N_n-1}|d_{jk}^{(2)n}| \leqslant Cp_n^{2+2\gamma}, \tag{23}$$

*for some constant $C$ and every sequence of partitions $(\pi_n)$ of the interval $[0,T]$, where $d_{jk}^{(2)n} = \mathbf{E}(\Delta_{ir,j}^{(2)n}X\Delta_{ir,k}^{(2)n}X)$, $1 \leqslant j,k \leqslant n$. Then*

$$V_{\pi_n}^{(2)}(X,2) \longrightarrow 2\kappa^2\int_0^T h(\ell(t))\,dt \quad a.s. \qquad as\ n\to\infty.$$

**Proof.** The proof is standard. We give it for completeness in Appendix. One can found it, i.e., in [2].

**Corollary 17** *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval $[0,T]$, $T > 0$. Assume that $X$ is a Gaussian process satisfying conditions $(C1)$ and $(C2)$ and having the Orey index $\gamma$. Moreover, assume that*

$$\max_{1\leqslant k\leqslant N_n-1}\sum_{j=1}^{N_n-1}|d_{jk}^{(2)n}| \leqslant C\Big(\frac{T}{N_n}\Big)^{2+2\gamma} \tag{24}$$

*for some constant $C$, and every sequence of partitions $(\pi_n)$ of the interval $[0,T]$, where $d_{jk}^{(2)n} = \mathbf{E}(\Delta_{n,j}^{(2)}X\Delta_{n,k}^{(2)}X)$, $1 \leqslant j,k \leqslant N_n-1$. Then*

$$V_{N_n}^{(2)}(X,2) \longrightarrow \kappa^2(4-2^{2\gamma})T \qquad a.s. \quad as\ n\to\infty.$$

**Proof.** For regular partition $\pi_n$ condition (23) transforms to (24).

**Theorem 18** *Assume that conditions of Proposition 15 are satisfied. Moreover, assume that inequality (24) holds, then*

$$V_{N_n}^{(2)}(X,2) \longrightarrow \int_0^T g_0(t)\,dt \qquad a.s. \quad as\ n\to\infty.$$

**Proof.** The proof of the theorem evidently follows from Proposition 15 and arguments used to prove Theorem 16.

**Remark 19** *Let $X$ be a sfBm and let $H \neq 1/2$. Then in assumption (21) the function $\varphi(h)$ we could not change by $h$ or $0$. Observe that following equality*

$$\begin{aligned}\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2 =&(4-2^{2H})h^{2H} - 2^{2H-1}(t+h)^{2H} - 3\cdot 2^{2H}t^{2H}\\ &- 2^{2H-1}(t-h)^{2H} + 2(2t+h)^{2H} + 2(2t-h)^{2H}\end{aligned}$$

*holds. Set*

$$\lambda_t(h) := \mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2 - (4-2^{2H})h^{2H}$$

*and note that $\lambda_t(0) = \lambda_t'(0) = \lambda_t''(0) = \lambda_t^{(3)}(0) = 0$. The Taylor formula yields*

$$\lambda_t(h) = \int_0^h \frac{(h-x)^3}{3!}\lambda_t^{(4)}(x)\,dx, \qquad \forall\ h\leqslant t\leqslant T-h,$$

*where*

$$\begin{aligned}\lambda_t^{(4)}(x) =&C_H\Big(2\big[(2t+x)^{2H-4}+(2t-x)^{2H-4}\big] - 2^{2H-1}\big[(t+x)^{2H-4}+(t-x)^{2H-4}\big]\Big),\\ C_H =&2H(2H-1)(2H-2)(2H-3).\end{aligned}$$

*Note that*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T-h}&\left|\frac{\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2}{h^{2H}}-(4-2^{2H})\right|\\&\geqslant \sup_{h\leqslant t\leqslant T-h}\left|\int_0^h \frac{(h-x)^3}{3!h^{2H}}\,\lambda_t^{(4)}(x)\,dx\right|\geqslant\left|\int_0^h \frac{(h-x)^3}{3!h^{2H}}\,\lambda_h^{(4)}(x)\,dx\right|.\end{aligned}$$

*After a change of variable $y=\frac{h-x}{ah+bx}$ with certain constants $a$ and $b$, we obtain equality*

$$\begin{aligned}h^{-2H}\int_0^h (h-x)^3\lambda_h^{(4)}(x)\,dx =&2\cdot 3^{2H}C_H\int_0^{1/2} y^3(1+y)^{-2H-1}dy+2C_H\int_0^{1/2} y^3(1-y)^{-2H-1}dy\\&+6^{2H}C_H\int_0^1 y^3(1+y)^{-2H-1}dy+2^{2H-2}H^{-1}C_H.\end{aligned}$$

*All these integrals are finite and don't depend on $h$. Thus*

$$\lim_{h\to 0+}\sup_{h\leqslant t\leqslant T-h}\left|\frac{\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2}{h^{2H}}-(4-2^{2H})\right|>0.$$

*For this reason the condition (c) of Theorem 1 in Bégyn [3] is not satisfied for a sfBm $X$ with $H\neq 1/2$. In the case under consideration condition (c) has the form*

$$\lim_{h\to 0+}\sup_{h\leqslant t\leqslant T-h}\left|\frac{(\delta_1^h\circ\delta_2^h R)(t,t)}{h^{2H}}-(4-2^{2H})\right|=0,$$

*where $R(s,t)$ is a covariance function of a sfBm and*

$$\begin{aligned}(\delta_1^h\circ\delta_2^h R)(t,t):=&4R(t,t)+2R(t-h,t+h)-4R(t+h,t)\\&-4R(t-h,t)+R(t+h,t+h)+R(t-h,t-h)\\=&\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2.\end{aligned}$$

*On the other hand, assumption (21) is satisfied for sfBm. Really, from inequality*

$$\begin{aligned}\sup_{\varphi(h)\leqslant t\leqslant T-h}&\left|\frac{\mathbf{E}\big(X_{t+h}-2X_t+X_{t-h}\big)^2}{h^{2H}}-(4-2^{2H})\right|\\&\leqslant h^{-2H}\sup_{\varphi(h)\leqslant t\leqslant T-h}\sup_{0\leqslant x\leqslant h}|\lambda_t^{(4)}(x)|\int_0^h (h-x)^3\,dx\\&\leqslant |C_H|\cdot h^{4-2H}\sup_{\varphi(h)\leqslant t\leqslant T-h}\left(\frac{2}{(2t)^{4-2H}}+\frac{2}{(2t-h)^{4-2H}}+\frac{2^{2H-1}}{t^{4-2H}}+\frac{2^{2H-1}}{(t-h)^{4-2H}}\right)\\&\leqslant |C_H|\cdot h^{4-2H}\left(\frac{2}{(2\varphi(h))^{4-2H}}+\frac{2}{(2\varphi(h)-h)^{4-2H}}+\frac{2^{2H-1}}{\varphi(h)^{4-2H}}+\frac{2^{2H-1}}{(\varphi(h)-h)^{4-2H}}\right)\\&\leqslant |C_H|\cdot\left[\left(\frac{h}{\varphi(h)}\right)^{4-2H}+\frac{2}{(2L(h)-1)^{4-2H}}+2^{2H-1}\left(\frac{h}{\varphi(h)}\right)^{4-2H}+\frac{2^{2H-1}}{(L(h)-1)^{4-2H}}\right]\end{aligned}$$

*we obtain the required assertion.*

## 3.1 Bifractional Brownian motion

We shall prove that the conditions of Theorem 16 are satisfied for bifBm. The bifBm satisfies conditions $(C1)$ and $(C2)$. So it suffices to verify the inequality (23).

Repeating outlines of the proof of Theorem 4 of Begyn [2] the study of the asymptotic properties of the $d_{jk}^{(2)n}$ we divide into three steps, according, to the value of $k-j$.

If $j=k$ then (15) yields

$$\begin{aligned}d_{kk}^{(2)n}\leqslant&2\big[(\Delta_k^n t)^2\mathbf{E}(\Delta_{k+1}^n B^{HK})^2+(\Delta_{k+1}^n t)^2\mathbf{E}(\Delta_k^n B^{HK})^2\big]\\\leqslant&2^{2-K}\big[(\Delta_k^n t)^2|t_{k+1}-t_k|^{2HK}+(\Delta_{n,k+1}t)^2|t_k-t_{k-1}|^{2HK}\big]\\\leqslant&2^{3-K}m_n^{2+2HK}.\end{aligned}\tag{25}$$

By using the Cauchy-Schwarz inequality we get

$$\left|d_{jk}^{(2)n}\right| \leqslant \mathbf{E}^{1/2}\left|(\Delta_{ir,j}^{(2)n} B^{HK})\right|^2 \cdot \mathbf{E}^{1/2}\left|(\Delta_{ir,k}^{(2)n} B^{HK})\right|^2 \leqslant 2^{3-K} m_n^{2+2HK} \tag{26}$$

for $1 \leqslant k-j \leqslant 2$ and

$$d_{j1}^{(2)n} \leqslant 2^{3-K} m_n^{2+2HK} \quad \text{for } 1 \leqslant j \leqslant N_n - 1, \tag{27}$$

$$d_{1k}^{(2)n} \leqslant 2^{3-K} m_n^{2+2HK} \quad \text{for } 1 \leqslant k \leqslant N_n - 1. \tag{28}$$

Now consider the case $|j-k| \geqslant 3$. By symmetry of $d_{jk}^{(2)n}$ one can take $j-k \geqslant 3$. Note that **for** $j \neq 1$ **and** $k \neq 1$ **equality**

$$d_{jk}^{(2)n} = \int_{t_j^n}^{t_{j+1}^n} du \int_{t_{j-1}^n}^{t_j^n} dv \int_v^u dw \int_{t_k^n}^{t_{k+1}^n} dx \int_{t_{k-1}^n}^{t_k^n} dy \int_y^x \frac{\partial^4 R_{HK}}{\partial s^2 \partial t^2}(w,z)\, dz$$

holds. The fourth order mixed partial derivative of the covariance function $R_{HK}(s,t)$ is of the following form

$$\begin{aligned}\frac{\partial^4 R_{HK}}{\partial s^2 \partial t^2}(s,t) =& -\frac{2HK(2H-1)(2HK-2)(2HK-3)}{2^K|s-t|^{2(2-KH)}} \\ &+ \frac{K(K-1)(K-2)(K-3)(2H)^4}{2^K}(st)^{4H-2}\left(s^{2H}+t^{2H}\right)^{K-4} \\ &+ \frac{K(K-1)(2H)^2(2H-1)}{2^K}\left[(K-2)(2H)+(2H-1)\right](st)^{2H-2}\left(s^{2H}+t^{2H}\right)^{K-2}\end{aligned}$$

for each $s,t>0$ such that $s \neq t$. Since $2s^H t^H \leqslant s^{2H}+t^{2H}$ and $K-2<0$, $K-4<0$ it follows that

$$\begin{aligned}(st)^{2H-2}\left(s^{2H}+t^{2H}\right)^{K-2} &\leqslant 2^{K-2}(st)^{KH-2} \\ (st)^{4H-2}\left(s^{2H}+t^{2H}\right)^{K-4} &\leqslant 2^{K-4}(st)^{KH-2}.\end{aligned}$$

Thus

$$\left|\frac{\partial^4 R^{HK}}{\partial s^2 \partial t^2}(s,t)\right| \leqslant \frac{C_1}{|s-t|^{2(2-KH)}} + \frac{C_2}{(st)^{2-KH}}$$

and

$$\begin{aligned}|d_{jk}^{(2)n}| \leqslant& \int_{t_j^n}^{t_{j+1}^n} du \int_{t_{j-1}^n}^{t_j^n} dv \int_v^u dw \int_{t_k^n}^{t_{k+1}^n} dx \int_{t_{k-1}^n}^{t_k^n} dy \int_y^x \frac{C_1}{|w-z|^{2(2-KH)}}\, dz \\ &+ \int_{t_j^n}^{t_{j+1}^n} du \int_{t_{j-1}^n}^{t_j^n} dv \int_v^u dw \int_{t_k^n}^{t_{k+1}^n} dx \int_{t_{k-1}^n}^{t_k^n} dy \int_y^x \frac{C_2}{(wz)^{2-KH}}\, dz \\ =:& I_{jk}^{n,1} + I_{jk}^{n,2},\end{aligned} \tag{29}$$

where constants $C_1$ and $C_2$ depends on $H$ and $K$. Inequality

$$|w-z| \geqslant t_{j-1}^n - t_{k+1}^n = \sum_{i=k+2}^{j-1} \Delta_{n,i} t \geqslant (j-k-2)p_n$$

on the integration set imply

$$I_{jk}^{n,1} \leqslant \frac{4C_1 m_n^6}{(j-k-2)^{2(2-HK)} p_n^{2(2-HK)}} \leqslant \frac{4C_1 c^6 p_n^{2+2HK}}{(j-k-2)^{2(2-HK)}}, \tag{30}$$

where $c$ is a constant defined in Definition 11. Moreover,

$$\sum_{j-k\geqslant 3}^{n-1} \frac{1}{(j-k-2)^{2(2-HK)}} \leqslant \sum_{j=1}^{\infty} \frac{1}{j^{2(2-KH)}} < \infty. \tag{31}$$

Now we estimate $I_{jk}^{n,2}$. By modifying the computations above we similarly find that

$$
\begin{aligned}
I_{jk}^{n,2} \leqslant & \frac{4C_2 m_n^6}{(t_{j-1}t_{k-1})^{2-KH}} = \frac{4C_2 m_n^6}{(t_{k-1}\sum_{i=k}^{j-1}\Delta_i t + t_{k-1}^2)^{2-KH}} \\
\leqslant & \frac{4C_2 m_n^6}{p_n^{2-KH}((t_{j-1}-t_{k-1})+t_{k-1})^{2-KH}} \leqslant \frac{4C_2 c^6 p_n^{4+KH}}{(t_{j-1}-t_{k-1})^{2-KH}} \\
\leqslant & 4C_2 c^6 \, \frac{p_n^{2+2KH}}{(j-k)^{2-KH}} . \qquad (32)
\end{aligned}
$$

Note that

$$
\sum_{j-k\geqslant 3}^{N_n-1} \frac{1}{(j-k)^{2-KH}} \leqslant \sum_{j=1}^{\infty} \frac{1}{j^{2-KH}} < \infty. \qquad (33)
$$

The inequality (23) follows from inequalities (29)-(33).

## 3.2 Subfractional Brownian motion

We recall that conditions $(C1)$ and $(C2)$ are satisfied for sfBm. So the statement of Theorem 16 is satisfied if inequality (23) holds. To prove this, we apply similar arguments as for bifBm.

If $j = k$ or $1 \leqslant k - j \leqslant 2$ then (13) and (14) yields

$$
d_{jk}^{(2)n} \leqslant 8m_n^{2+2H}.
$$

The same inequality holds for $d_{j1}^{(2)n}$, $1 \leqslant j \leqslant N_n - 1$ and $d_{1k}^{(2)n}$, $1 \leqslant k \leqslant N_n - 1$.

The fourth order mixed partial derivative of the covariance function $G_H(s,t)$ is of the following form

$$
\frac{\partial^4 G_H}{\partial s^2 \partial t^2}(s,t) = -H(2H-1)(2H-2)(2H-3)\left[\frac{1}{|s-t|^{2(2-H)}} + \frac{1}{(s+t)^{2(2-H)}}\right].
$$

for each $s, t > 0$ such that $s \neq t$. Note that $(s+t)^{2(2-H)} \geqslant |s-t|^{2(2-H)}$ if $s \neq t$.

Thus

$$
\left|\frac{\partial^4 G_H}{\partial s^2 \partial t^2}(s,t)\right| \leqslant \frac{2H(2H-1)(2H-2)(2H-3)}{|s-t|^{2(2-H)}}
$$

and

$$
|d_{jk}^{(2)n}| \leqslant \frac{4C_H m_n^6}{(j-k-2)^{2(2-H)} p_n^{2(2-H)}} \leqslant \frac{4C_H c^6 p_n^{2+2H}}{(j-k-2)^{2(2-H)}}
$$

for $j - k \geqslant 3$, $2 \leqslant k \leqslant N_n - 1$, where $C_H = 2H(2H-1)(2H-2)(2H-3)$, $c$ is a constant defined in Definition 11. Thus

$$
\begin{aligned}
\max_{2\leqslant k\leqslant N_n-1} \sum_{j-k\geqslant 3} d_{jk}^{(2)n} \leqslant & 4C_H c^6 p_n^{2+2H} \max_{2\leqslant k\leqslant N_n-1} \sum_{j-k\geqslant 3} \frac{1}{(j-k-2)^{2(2-H)}} \\
\leqslant & 4C_H c^6 p_n^{2+2H} \sum_{j=1}^{\infty} \frac{1}{j^{2(2-H)}} \leqslant C p_n^{2+2H} \qquad (34)
\end{aligned}
$$

for some constant $C$ and inequality (23) holds.

## 3.3 Ornstein-Uhlenbeck process

First we show the following lemma.

**Lemma 20** *Let $X$ be the solution of equation (16). Then*

$$
\left|V_{\pi_n}^{(2)}(X,2) - \theta^2 V_{\pi_n}^{(2)}(B^H,2)\right| = O(p_n^{1-\varepsilon})
$$

*for every $\varepsilon > 0$.*

**Proof.** It is evident that

$$\Delta_{ir,k}^{(2)n} X = -\mu\left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} X_s\, ds - \Delta_{k+1}^n t \int_{t_{k-1}^n}^{t_k^n} X_s\, ds\right) + \theta \Delta_{ir,k}^{(2)n} B^H.$$

After simple calculations we get the estimate

$$\begin{aligned}\sup_{t_k^n \leqslant s \leqslant t_{k+1}^n} |X_s - X_k| &\leqslant \mu(\Delta_{k+1}^n t) \sup_{t \leqslant T} |X_t| + \theta \sup_{t_k^n \leqslant s \leqslant t_{k+1}^n} (B_s^H - B_k^H) \\ &\leqslant \mu\, m_n \sup_{t \leqslant T} |X_t| + \theta L_T^{H, H-\varepsilon} m_n^{H-\varepsilon},\end{aligned}$$

where $L_T^{H,H-\varepsilon}$ is defined in subsection 2.3. Thus

$$\begin{aligned}&\left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} (X_s - X_k)\, ds - \Delta_{k+1}^n t \int_{t_{k-1}^n}^{t_k^n} (X_s - X_k)\, ds\right)^2 \\ &\quad\leqslant 2m_n^3 \int_{t_k^n}^{t_{k+1}^n} (X_s - X_k)^2\, ds + 2m_n^3 \int_{t_{k-1}^n}^{t_k^n} (X_s - X_k)^2\, ds \\ &\quad\leqslant 2m_n^4 \Big(\sup_{t_k^n \leqslant s \leqslant t_{k+1}^n} (X_s - X_k)^2 + \sup_{t_{k-1}^n \leqslant s \leqslant t_k^n} (X_k - X_s)^2\Big) \\ &\quad\leqslant 8m_n^{4+2H-2\varepsilon} \Big(\mu^2 m_n^{2-2H+2\varepsilon} \sup_{t \leqslant T} X_t^2 + \theta^2 (L_T^{H,H-\varepsilon})^2\Big)\end{aligned}$$

and

$$\begin{aligned}&\left|\left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} X_s\, ds - \Delta_{k+1}^n t \int_{t_{k-1}^n}^{t_k^n} X_s\, ds\right) \Delta_{ir,k}^{(2)n} B^H\right| \\ &\quad= \left|\left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} (X_s - X_k)\, ds - \Delta_{n,k+1} t \int_{t_{k-1}^n}^{t_k^n} (X_s - X_k)\, ds\right) \Delta_{ir,k}^{(2)n} B^H\right| \\ &\quad\leqslant 2m_n^{2+H-\varepsilon} \Big(\mu\, m_n^{1-H+\varepsilon} \sup_{t \leqslant T} |X_t| + \theta L_T^{H,H-\varepsilon}\Big) \cdot 2m_n L_T^{H,H-\varepsilon} m_n^{H-\varepsilon} \\ &\quad= 4m_n^{3+2H-2\varepsilon} \Big(\mu\, m_n^{1-H+\varepsilon} \sup_{t \leqslant T} |X_t| + \theta L_T^{H,H-\varepsilon}\Big) \cdot L_T^{H,H-\varepsilon}.\end{aligned}$$

Thus for every $\varepsilon > 0$

$$\begin{aligned}\left|V_{\pi_n}^{(2)}(X, 2) - \theta^2 V_{\pi_n}^{(2)}(B^H, 2)\right| \leqslant & 8c^{2+2H} m_n^{2-2\varepsilon} \Big(\mu^2 m_n^{2-2H+2\varepsilon} \sup_{t \leqslant T} X_t^2 + 2\theta^2 (L_T^{H,H-\varepsilon})^2\Big) T \\ & + 4c^{2+2H} m_n^{1-2\varepsilon} \Big(\mu m_n^{1-H+\varepsilon} \sup_{t \leqslant T} |X_t| + \theta L_T^{H,H-\varepsilon}\Big) \cdot L_T^{H,H-\varepsilon} T \\ = & O(m_n^{1-2\varepsilon})\end{aligned}$$

since

$$\frac{1}{\mu_k^n} \leqslant \frac{1}{2p_n^{2H+2}}.$$

This implies the statement of the lemma. ■

As in previous cases it is enough to verify condition (23) of Theorem 16 for fBm $B^H$. The following inequality

$$\left|\frac{\partial^4 F_H}{\partial s^2 \partial t^2}(s, t)\right| \leqslant \frac{H|(2H-1)(2H-2)(2H-3)|}{|s-t|^{2(2-H)}},$$

holds for the covariance function $F_H(s, t)$ of $B^H$. Applying similar arguments as for sfBm we obtain

$$\max_{1 \leqslant k \leqslant N_n - 1} \sum_{j=1}^{N_n - 1} |d_{jk}^{(2)n}| \leqslant C p_n^{2+2H}.$$

From Lemma 20 and inequality above we get the statement of Theorem 16.

## 3.4 Fractional Brownian bridge

The fractional Brownian bridge is defined in $[0,T]$ by

$$X_t^H = B_t^H - \frac{t^{2H}+T^{2H}-|t-T|^{2H}}{2T^{2H}}\, B_T^H,$$

where $B^H$ is a fBm in $[0,T]$. Denote $g(t,T)=t^{2H}+T^{2H}-|t-T|^{2H}$. Then

$$\begin{aligned}
\mathbf{E}X_t^H X_s^H =&\mathbf{E}B_t^H B_s^H-(2T^{2H})^{-1}g(t,T)\mathbf{E}B_T^H B_s^H-(2T^{2H})^{-1}g(s,T)\mathbf{E}B_T^H B_t^H\\
&+(2T^{2H})^{-2}g(s,T)g(t,T)\mathbf{E}(B_T^H)^2\\
=&\mathbf{E}B_t^H B_s^H-(2T^{2H})^{-1}g(s,T)g(t,T)+(4T^{2H})^{-1}g(s,T)g(t,T)\\
=&\mathbf{E}B_t^H B_s^H-(4T^{2H})^{-1}g(s,T)g(t,T).
\end{aligned}$$

If $j=k$ or $1\leqslant k-j\leqslant 2$ then application of inequality (17) yields

$$d_{jk}^{(2)n}\leqslant 4m_n^{2+2H}.$$

The same inequality holds for $d_{j1}^{(2)n}$, $1\leqslant j\leqslant N_n-1$ and $d_{1k}^{(2)n}$, $1\leqslant k\leqslant N_n-1$.

Further

$$\begin{aligned}
\left|\frac{\partial^4 g(s,T)g(t,T)}{\partial s^2\partial t^2}\right| =&4H^2(2H-1)^2\big|\big(s^{2H-2}-(T-s)^{2H-2}\big)\big(t^{2H-2}-(T-t)^{2H-2}\big)\big|\\
\leqslant&4H^2(2H-1)^2\big[(st)^{2H-2}+[t(T-s)]^{2H-2}+[s(T-t)]^{2H-2}\\
&+[(T-t)(T-s)]^{2H-2}\big].
\end{aligned}$$

Similarly as for bifBm we estimate $d_{jk}^{(2)n}$. Assume that $|j-k|\geqslant 3$. By symmetry of $d_{jk}^{(2)n}$ one can take $j-k\geqslant 3$. We first note that

$$\begin{aligned}
I_1:=&\int_{t_j^n}^{t_{j+1}^n}du\int_{t_{j-1}^n}^{t_j^n}dv\int_v^u dw\int_{t_k^n}^{t_{k+1}^n}dx\int_{t_{k-1}^n}^{t_k^n}dy\int_y^x\frac{1}{(wz)^{2-2H}}\,dz\\
\leqslant&\frac{4m_n^6}{(t_{j-1}^n t_{k-1}^n)^{2-2H}}\leqslant 4c^6\,\frac{p_n^{2+4H}}{(j-k)^{2-2H}}\,,
\end{aligned}$$

Assume $k\neq N_n-1$ and $j\neq 1$ or $j\neq N_n-1$ and $k\neq 1$. Then

$$\begin{aligned}
I_2:=&\int_{t_j^n}^{t_{j+1}^n}du\int_{t_{j-1}^n}^{t_j^n}dv\int_v^u dw\int_{t_k^n}^{t_{k+1}^n}dx\int_{t_{k-1}^n}^{t_k^n}dy\int_y^x\frac{1}{[w(T-z)]^{2-2H}}\,dz\\
\leqslant&\frac{4m_n^6}{[t_{j-1}^n(t_{j-1}^n-t_{k+1}^n)]^{2-2H}}\leqslant\frac{4m_n^6}{p_n^{2-2H}(t_{j-1}^n-t_{k+1}^n)^{2-2H}}\leqslant 4c^6\,\frac{p_n^{2+4H}}{(j-k-2)^{2-2H}}\,,\\
I_3:=&\int_{t_j^n}^{t_{j+1}^n}du\int_{t_{j-1}^n}^{t_j^n}dv\int_v^u dw\int_{t_k^n}^{t_{k+1}^n}dx\int_{t_{k-1}^n}^{t_k^n}dy\int_y^x\frac{1}{[z(T-w)]^{2-2H}}\,dz\\
\leqslant&\frac{4m_n^6}{[t_{k-1}^n(T-t_{j+1}^n)]^{2-2H}}\leqslant\frac{4m_n^6}{p_n^{2-2H}(T-t_{j+1}^n)^{2-2H}}\leqslant 4c^6\,\frac{p_n^{2+4H}}{(N_n-j-1)^{2-2H}},\\
I_4:=&\int_{t_j^n}^{t_{j+1}^n}du\int_{t_{j-1}^n}^{t_j^n}dv\int_v^u dw\int_{t_k^n}^{t_{k+1}^n}dx\int_{t_{k-1}^n}^{t_k^n}dy\int_y^x\frac{1}{[(T-w)(T-z)]^{2-2H}}\,dz\\
\leqslant&\frac{4m_n^6}{p_n^{2-2H}(t_{j+1}^n-t_{k+1}^n)^{2-2H}}\leqslant 4c^6\,\frac{p_n^{2+4H}}{(j-k)^{2-2H}}\,.
\end{aligned}$$

Thus we obtain

$$\max_{1\leqslant k\leqslant N_n-1}\sum_{j=1}^{N_n-1}|d_{jk}^{(2)n}|\leqslant Cp_n^{2+2H}.$$

and get the statement of Theorem 16.

# 4 On the estimation of Orey index for irregular partition

Let $(\pi_n)_{n\geqslant 1}$ be a sequence of partitions of $[0,T]$ such that $0=t_0^n<t_1^n<\cdots<t_{m(n)}^n=T$ for all $n\geqslant 1$. Assume that we have two sequences of partitions $(\pi_{i(n)})_{n\geqslant 1}$ and $(\pi_{j(n)})_{n\geqslant 1}$ of $[0,T]$ such that $\pi_{i(n)}\subset\pi_{j(n)}\subseteq\pi_n$, $i(n)<j(n)\leqslant m(n)$, for all $n\in\mathbb{N}$, where $\pi_{i(n)}=\{0=t_0^n<t_{i(1)}^n<t_{i(2)}^n<\cdots<t_{i(n)}^n=T\}$ and $\pi_{j(n)}=\{0=t_0^n<t_{j(1)}^n<t_{j(2)}^n<\cdots<t_{j(n)}^n=T\}$. Set

$$\Delta_{i(k)}^n t=t_{i(k)}^n-t_{i(k-1)}^n,\qquad m_{i(n)}=\max_{1\leqslant k\leqslant i(n)}\Delta_{i(k)}^n t,\qquad p_{i(n)}=\min_{1\leqslant k\leqslant i(n)}\Delta_{i(k)}^n t.$$

Moreover, assume that $p_{j(n)}\neq m_{i(n)}$ and $m_{i(n)}\leqslant cp_{i(n)}$, for all $i(n)$, $n\geqslant 1$, $c\geqslant 1$. Note that $p_{j(n)}\leqslant p_{i(n)}$.

Let $X$ be a Gaussian process with Orey index $\gamma\in(0,1)$. Set

$$V_{\pi_{i(n)}}^{(2)}(X,2)=2\sum_{k=1}^{i(n)-1}\frac{\Delta_{i(k+1)}^n t(\Delta_{ir,k}^{(2)n}X)^2}{(\Delta_{i(k)}^n t)^{\gamma+1/2}(\Delta_{i(k+1)}^n t)^{\gamma+1/2}[\Delta_{i(k)}^n t+\Delta_{i(k+1)}^n t]},$$

where

$$\Delta_{ir,i(k)}^{(2)n}X=\Delta_{i(k)}^n t\cdot X(t_{i(k+1)}^n)+\Delta_{i(k+1)}^n t\cdot X(t_{i(k-1)}^n)-(\Delta_{i(k)}^n t+\Delta_{i(k+1)}^n t)X(t_{i(k)}^n).$$

Denote

$$V_{i(n)}^{(2)}(X,2)=\sum_{k=1}^{i(n)-1}(\Delta_{ir,k}^{(2)n}X)^2\quad\text{and}\quad \mu_k^n=(\Delta t_{i(k)}^n)^{\gamma+1/2}(\Delta_{i(k)}^n t)^{\gamma+1/2}[\Delta_{i(k)}^n t+\Delta_{i(k+1)}^n t].$$

Define

$$\widehat{\gamma}_n=-\frac{1}{2}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ln\frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)}.$$

**Theorem 21** *Assume that conditions of Proposition 13 are satisfied for two sequences of partitions $(\pi_{i(n)})_{n\geqslant 1}$ and $(\pi_{j(n)})_{n\geqslant 1}$ of $[0,T]$ with the properties mentioned above. Then*

$$V_{\pi_{k(n)}}^{(2)}(X,2)\longrightarrow 2\kappa^2\int_0^T h(\ell(t))\,dt\quad a.s.\qquad as\ n\to\infty \tag{35}$$

*for $k(n)=i(n)$ or $k(n)=j(n)$. If sequences of partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, $i(n)<j(n)$, are regular or such that $p_{j(n)}/p_{i(n)}\to 0$ as $n\to\infty$, then*

$$\widehat{\gamma}_n\xrightarrow{\mathbf{a.s.}}\gamma.$$

**Proof**. Proposition 13 yields the limit (35). It is evident that

$$\frac{1}{2m_n^{2\gamma+1}}\leqslant\frac{\Delta t_{i(k)}^n}{\mu_k^n}\leqslant\frac{1}{2p_n^{2\gamma+1}}$$

and

$$\left(\frac{p_{i(n)}}{m_{j(n)}}\right)^{2\gamma+1}\frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)}\leqslant\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\leqslant\left(\frac{m_{i(n)}}{p_{j(n)}}\right)^{2\gamma+1}\frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)}. \tag{36}$$

Further

$$\begin{aligned}
\widehat{\gamma}_n=&-\frac{1}{2}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\left((2\gamma+1)\ln(p_{j(n)}/m_{i(n)})+\ln\frac{m_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{p_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\right)\\
=&\gamma+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ln\frac{m_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{p_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\\
=&\gamma+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\\
&+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ln\left(\frac{m_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{p_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right)\\
\leqslant&\gamma+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}
\end{aligned}$$

since $\ln(p_{j(n)}/m_{i(n)}) \leqslant 0$

$$\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\left(\frac{m_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{p_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right)\leqslant 0$$

In the same way we get

$$\begin{aligned}
\widehat{\gamma}_n =& -\frac{1}{2}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\left((2\gamma+1)\ln(m_{j(n)}/p_{i(n)})+\ln\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\right)\\
=& -\frac{1}{2}+\left(\gamma+\frac{1}{2}\right)\frac{\ln(m_{j(n)}/p_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\\
=&\gamma+\left(\gamma+\frac{1}{2}\right)\frac{\ln(m_{j(n)}/p_{i(n)})-\ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\\
&+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\left(\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right)\\
\geqslant&\gamma+\left(\gamma+\frac{1}{2}\right)\frac{\ln(m_{j(n)}/p_{i(n)})-\ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\,,
\end{aligned}\tag{37}$$

since

$$\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\left(\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right)\geqslant 0$$

and

$$\left(\gamma+\frac{1}{2}\right)\frac{\ln(m_{j(n)}/p_{i(n)})-\ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})}\leqslant 0.$$

If sequences of partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, $i(n)<j(n)$, are regular then the second term in the inequality (37) is equal to 0 and

$$|\widehat{\gamma}^n-\gamma|\leqslant\frac{1}{2\ln(m_{i(n)}/p_{j(n)})}\left|\ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right|.$$

Under conditions of the theorem in the regular case of partitions the statement of the theorem hold. In an irregular case of partitions we obtain inequalities

$$\left|\widehat{\gamma}_n-\gamma-\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right|\leqslant\left(\gamma+\frac{1}{2}\right)\frac{\ln(m_{j(n)}/p_{j(n)})+\ln(m_{i(n)}/p_{i(n)})}{\ln(m_{i(n)}/p_{j(n)})}$$

and

$$\begin{aligned}
|\widehat{\gamma}^n-\gamma|\leqslant&\left|\widehat{\gamma}^n-\gamma-\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}+\frac{1}{2\ln(p_{j(n)}/m_{i(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right|\\
\leqslant&\frac{3}{2}\,\frac{\ln(m_{j(n)}/p_{j(n)})+\ln(m_{i(n)}/p_{i(n)})}{\ln(m_{i(n)}/p_{j(n)})}+\frac{1}{2\ln(m_{i(n)}/p_{j(n)})}\left|\ln\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right|.
\end{aligned}$$

For irregular case of partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, $i(n)<j(n)$, the second term in above inequality goes to 0 as $\ln(p_{i(n)}/p_{j(n)})\to\infty$, $n\to\infty$. Thus the statement of the theorem holds.

# 5 Appendix

## 5.1 Proof of Lemma 4

Assume, without lost of generality, that $0<h<1$. We first prove that $\widehat{\gamma}_*\leqslant\gamma_*$, where

$$\widehat{\gamma}_*:=\limsup_{h\downarrow 0}\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h}\,,\quad \gamma_*:=\inf\left\{\gamma>0\colon\ \lim_{h\downarrow 0}\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^\gamma}{\sigma_X(s,s+h)}=0\right\}.$$

Let $\gamma > \gamma_*$. It suffices to show that $\gamma \geqslant \widehat{\gamma}_*$. By definition of the greatest lower bound, there exists a real number $\alpha$ such that $\gamma > \alpha > \gamma_*$, and

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\alpha}{\sigma_X(s,s+h)} \longrightarrow 0 \quad \text{as } h\downarrow 0.$$

But

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} = h^{\gamma-\alpha} \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\alpha}{\sigma_X(s,s+h)} \longrightarrow 0 \quad \text{as } h\downarrow 0 \tag{38}$$

as the product of two functions tending to 0. Under the statement

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h) \longrightarrow 0 \qquad \text{as } h\downarrow 0 \tag{39}$$

and relation (38) there exists an $h_0$ such that for all $h \leqslant h_0 < 1$

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} = \frac{h^\gamma}{\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)} < 1 \quad \text{and} \quad \sup_{0\leqslant s\leqslant T-h}\sigma_X(s,s+h) < 1.$$

Moreover,

$$h^\gamma < \inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)$$

for all $h \leqslant h_0 < 1$. So

$$\ln h^\gamma < \ln\Big(\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)\Big) \leqslant \ln\Big(\sup_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)\Big)$$

and

$$\begin{aligned}\gamma > & \frac{\ln\big(\sup_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)\big)}{\ln h} = \sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h} \\ & \geqslant \limsup_{h\downarrow 0}\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h} = \widehat{\gamma}_*\,.\end{aligned}$$

Thus $\widehat{\gamma}_* \leqslant \gamma_*$.

Next we prove that $\widehat{\gamma}_* \geqslant \gamma_*$. Let $\gamma > \alpha > \widehat{\gamma}_*$. It suffices to show that $\gamma \geqslant \gamma_*$. Under the condition $\alpha > \widehat{\gamma}_*$ and statement (39) there exists $h_0$ such that for $h \leqslant h_0 < 1$

$$\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h} < \alpha, \qquad \sup_{0\leqslant s\leqslant T-h}\sigma_X(s,s+h) < 1.$$

This implies the inequality

$$\ln\Big(\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)\Big) > \ln h^\alpha$$

and

$$\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h) > h^\alpha.$$

Thus

$$\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^\alpha}{\sigma_X(s,s+h)} < 1.$$

So

$$\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^\gamma}{\sigma_X(s,s+h)} < h^{\gamma-\alpha} \longrightarrow 0 \quad \text{as } h\to 0.$$

Therefore $\gamma \geqslant \gamma_*$.

Now we prove that $\widehat{\gamma}^* = \gamma^*$, where

$$\widehat{\gamma}^* := \liminf_{h\downarrow 0}\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h}, \quad \gamma^* := \sup\left\{\gamma>0\colon\ \lim_{h\downarrow 0}\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^\gamma}{\sigma_X(s,s+h)} = +\infty\right\}.$$

We first prove $\widehat{\gamma}^* \geqslant \gamma^*$. By definition of greatest upper bound, there exists a real number $\gamma$ such that $\gamma^* > \gamma$, and

$$\lim_{h\downarrow 0}\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^\gamma}{\sigma_X(s,s+h)} = +\infty \tag{40}$$

It suffices to show that $\widehat{\gamma}^* \geqslant \gamma$. Under the condition $\gamma^* > \gamma$ and statements (39)-(40) there exists $h_0$ such that for $h \leqslant h_0 < 1$

$$\inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} > 1, \qquad \sup_{0\leqslant s\leqslant T-h} \sigma_X(s,s+h) < 1.$$

Moreover,

$$h^\gamma > \sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h) \geqslant \inf_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h)$$

and

$$\gamma \ln h > \ln \inf_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h), \qquad \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} > \gamma.$$

So $\widehat{\gamma}^* \geqslant \gamma$.

We show that $\gamma^* \geqslant \widehat{\gamma}^*$. Assume that $\widehat{\gamma}^* > \alpha > \gamma$. It sufficient to show that $\gamma^* > \gamma$. Under the condition $\widehat{\gamma}^* > \alpha$ and statement (39) there exists $h_0$ such that for $h \leqslant h_0 < 1$

$$\inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} > \alpha, \qquad \sup_{0\leqslant s\leqslant T-h} \sigma_X(s,s+h) < 1.$$

Moreover,

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} > \alpha$$

and

$$\ln\Big(\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h)\Big) < \ln h^\alpha.$$

Thus

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h) < h^\alpha \quad \text{and} \quad \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\alpha}{\sigma_X(s,s+h)} > 1.$$

Then

$$\inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} > h^{\gamma-\alpha} \to \infty$$

and $\gamma^* > \gamma$.

## 5.2 Proof of Theorem 16

Note that $V_{\pi_n}^{(2)}\big(X,2\big)$ is the square of the Euclidean norm of $(N_n-1)$-dimensional Gaussian vector which components are

$$\sqrt{\frac{2\Delta_k^n t}{\mu_k^n}}\,\Delta_{ir,k}^{(2)n} X, \qquad 1 \leqslant k \leqslant N_n - 1.$$

Denote by $(\lambda_{1,n},\dots,\lambda_{N_n-1,n})$ the eigenvalues of the corresponding covariance matrix whereas $\lambda_n^*$ stands for a maximal eigenvalue. There exists one $(N_n-1)$-dimensional Gaussian vector $Y_n$, such that its components are independent Gaussian variables $\mathcal{N}(0,1)$ and

$$V_{\pi_n}^{(2)}\big(X,2\big) = \sum_{j=1}^{N_n-1} \lambda_{j,n}\big(Y_n^{(j)}\big)^2, \qquad \mathbf{E}V_{\pi_n}^{(2)}\big(X,2\big) = \sum_{j=1}^{N_n-1} \lambda_{j,n}.$$

Since $\mathbf{E}V_{\pi_n}^{(2)}\big(X,2\big)$ is a convergent sequence (Proposition 13), the sums $\sum_{j=1}^{N_n-1}\lambda_{j,n}$ are bounded. Moreover, one has

$$\sum_{j=1}^{N_n-1} \lambda_{j,n}^2 \leqslant \lambda_n^* \sum_{j=1}^{N_n-1} \lambda_{j,n}.$$

From result of linear algebra and (23) we may further conclude that

$$\begin{aligned}
\lambda_n^* \leqslant& 2 \max_{1\leqslant k\leqslant N_n-1} \sum_{j=1}^{N_n-1} \sqrt{\frac{\Delta_{n,j}t\Delta_{n,k}t}{\mu_j^n\mu_k^n}}\,\big|\mathbf{E}\big(\Delta_{ir,j}^{(2)n}X\Delta_{ir,k}^{(2)n}X\big)\big| \\
\leqslant& 2\,\frac{1}{p_n^{2\gamma+1}} \max_{1\leqslant k\leqslant N_n-1} \sum_{j=1}^{N_n-1} |d_{jk}^{(2)n}| \leqslant Cp_n.
\end{aligned}$$

The Hanson and Wright's inequality (see [6], [2]) yields that for $\varepsilon > 0$

$$\mathbf{P}\left(\left|V_{\pi_n}^{(2)}(X,2) - \mathbf{E}V_{\pi_n}^{(2)}(X,2)\right| \geqslant \varepsilon\right) \leqslant 2\exp\left(-\min\left[\frac{C_1\varepsilon}{\lambda_n^*}, \frac{C_2\varepsilon^2}{\sum_{j=1}^{N_n-1}\lambda_{j,n}^2}\right]\right), \tag{41}$$

where $C_1$, $C_2$ are nonnegative constants. Therefore, the inequality (41) becomes

$$\mathbf{P}\left(\left|V_{\pi_n}^{(2)}(X,2) - \mathbf{E}V_{\pi_n}^{(2)}(X,2)\right| \geqslant \varepsilon\right) \leqslant 2\exp\left(-\frac{K\varepsilon^2}{\lambda_n^*}\right), \quad \forall\, 0 < \varepsilon \leqslant 1,$$

where $K$ is a positive constant. Set

$$\varepsilon_n^2 = \frac{2C}{K}\, p_n \ln n.$$

Then

$$\mathbf{P}\left(\left|V_{\pi_n}^{(2)}(X,2) - \mathbf{E}V_{\pi_n}^{(2)}(X,2)\right| \geqslant \varepsilon_n\right) \leqslant 2\exp\left(-2\ln n\right) = \frac{2}{n^2}\,.$$

Since $\lim_{n\to\infty}\varepsilon_n = 0$ and

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|V_{\pi_n}^{(2)}(X,2) - \mathbf{E}V_{\pi_n}^{(2)}(X,2)| \geqslant \varepsilon_n\right) < \infty$$

then Borel-Cantelli lemma gives the statement of the theorem.

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